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13. Ideal Quantum Gases I: Bosons

Gerhard Müller
University of Rhode Island, gmuller@uri.edu

Abstract
Part thirteen of course materials for Statistical Physics I: PHY525, taught by Gerhard Müller at the University of Rhode Island. Documents will be updated periodically as more entries become presentable.

Contents of this Document [ttc13]

Bose–Einstein functions [tsl36]

$$g_n(z) \equiv \frac{1}{\Gamma(n)} \int_0^\infty \frac{dx\, x^{n-1}}{z^{-1}e^x - 1} = \sum_{l=1}^\infty \frac{z^l}{l^n}, \qquad 0 \leq z \leq 1.$$

Special cases:

$$g_0(z) = \frac{z}{1-z}, \quad g_1(z) = -\ln(1-z), \quad g_\infty(z) = z.$$

Riemann zeta function:

$$g_n(1) = \zeta(n) \doteq \sum_{l=1}^\infty \frac{1}{l^n}.$$

Special values:

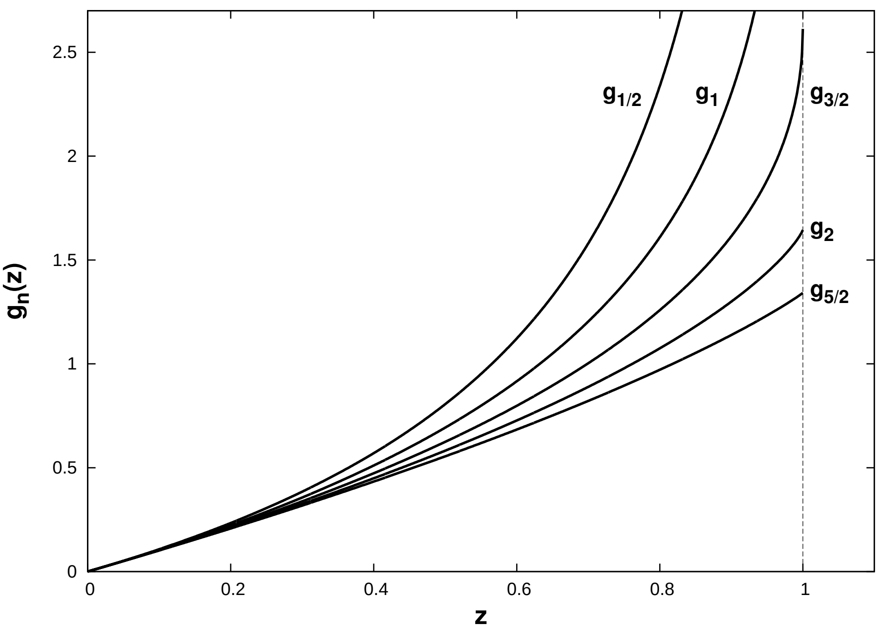
$$\zeta(1) \to \infty, \quad \zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}.$$

Recurrence relation:

$$z g_n'(z) = g_{n-1}(z), \qquad n \geq 1.$$

Singularity at $z = 1$ for non-integer n:

$$g_n(\alpha) = \Gamma(1-n)\alpha^{n-1} + \sum_{\ell=0}^\infty \frac{(-1)^\ell}{\ell!}\zeta(n-\ell)\alpha^\ell, \qquad \alpha \doteq -\ln z.$$



Ideal Bose-Einstein gas: equation of state and internal energy [tln67]

Conversion of sums into integrals by means of density of energy levels [tex113]:

$$D(\epsilon) = \frac{V}{\Gamma(\mathcal{D}/2)} \left(\frac{m}{2\pi\hbar^2}\right)^{\mathcal{D}/2} \epsilon^{\mathcal{D}/2-1}, \quad V = L^{\mathcal{D}}.$$

Fundamental thermodynamic relations for BE gas:

$$\frac{pV}{k_BT} = -\sum_k \ln\left(1 - ze^{-\beta\epsilon_k}\right) = -\int_0^\infty d\epsilon\, D(\epsilon) \ln\left(1 - ze^{-\beta\epsilon}\right) = \frac{V}{\lambda_T^{\mathcal{D}}} g_{\mathcal{D}/2+1}(z),$$

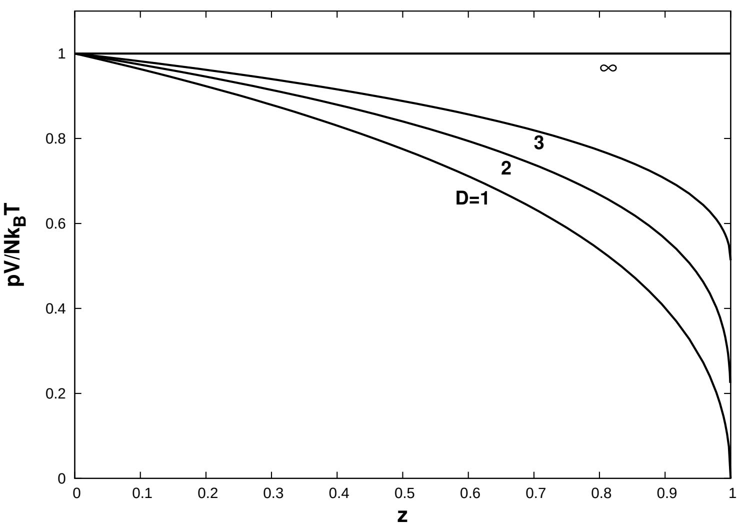
$$\mathcal{N} = \sum_k \frac{1}{z^{-1}e^{\beta\epsilon_k} - 1} = \int_0^\infty d\epsilon\, \frac{D(\epsilon)}{z^{-1}e^{\beta\epsilon} - 1} = \frac{V}{\lambda_T^{\mathcal{D}}} g_{\mathcal{D}/2}(z), \quad z < 1,$$

$$U = \sum_k \frac{\epsilon_k}{z^{-1}e^{\beta\epsilon_k} - 1} = \int_0^\infty d\epsilon\, \frac{D(\epsilon)\epsilon}{z^{-1}e^{\beta\epsilon} - 1} = \frac{\mathcal{D}}{2} k_BT \frac{V}{\lambda_T^{\mathcal{D}}} g_{\mathcal{D}/2+1}(z).$$

Warning: The range of fugacity is limited to the interval $0 \leq z \leq 1$. At $z = 1$, the expression for $\mathcal{N}$ must be amended by an additive term $z/(1-z)$ to account for the possibility of a macroscopic population of the lowest energy level (at $\epsilon = 0$). This amendment is only necessary for dimensionalities $\mathcal{D} > 2$, i.e. for the cases with $\lim_{\epsilon\to 0} D(\epsilon) = 0$.

Equation of state (with fugacity z in the role of parameter):

$$\frac{pV}{\mathcal{N}k_BT} = \frac{g_{\mathcal{D}/2+1}(z)}{g_{\mathcal{D}/2}(z)}, \quad z < 1.$$



[tex113] BE gas in $\mathcal{D}$ dimensions I: fundamental relations

From the expressions for the grand potential and the density of energy levels of an ideal Bose-Einstein gas in $\mathcal{D}$ dimensions and confined to a box of volume $V = L^{\mathcal{D}}$ with rigid walls,

$$\Omega(T,V,\mu) = k_BT\sum_k \ln(1 - ze^{-\beta\epsilon_k}), \qquad D(\epsilon) = \frac{V}{\Gamma(\mathcal{D}/2)}\left(\frac{m}{2\pi\hbar^2}\right)^{\mathcal{D}/2}\epsilon^{\mathcal{D}/2-1},$$

derive the fundamental thermodynamic relations at fugacity $z < 1$ in terms of the Bose-Einstein functions $g_n(z)$ and the thermal wavelength $\lambda_T = \sqrt{h^2/2\pi m k_BT}$ as follows:

$$\frac{pV}{k_BT} = \frac{V}{\lambda_T^{\mathcal{D}}}g_{\mathcal{D}/2+1}(z), \quad \mathcal{N} = \frac{V}{\lambda_T^{\mathcal{D}}}g_{\mathcal{D}/2}(z), \quad U = \frac{\mathcal{D}}{2}k_BT\frac{V}{\lambda_T^{\mathcal{D}}}g_{\mathcal{D}/2+1}(z).$$

Solution:

Reference Values for T, $V/\mathcal{N}$, and p [tln71]

The reference values introduced here are based on

(i) thermal wavelength: $\lambda_T \doteq \sqrt{\dfrac{h^2}{2\pi m k_B T}} = \sqrt{\dfrac{\Lambda}{k_B T}}, \quad \Lambda = \dfrac{h^2}{2\pi m}.$

(ii) MB equation of state: $pv = k_B T, \quad v \doteq V/\mathcal{N}.$

The reference values for k_BT, v, and p in isochoric, isothermal, and isobaric processes are

$$k_BT_v = \frac{\Lambda}{v^{2/\mathcal{D}}} \qquad p_v = \frac{\Lambda}{v^{2/\mathcal{D}+1}} \qquad (v = \text{const.})$$

$$v_T = \left(\frac{\Lambda}{k_BT}\right)^{\mathcal{D}/2} \qquad p_T = \Lambda\left(\frac{k_BT}{\Lambda}\right)^{\mathcal{D}/2+1} \qquad (T = \text{const.})$$

$$k_BT_p = \Lambda\left(\frac{p}{\Lambda}\right)^{2/(\mathcal{D}+2)} \qquad v_p = \left(\frac{\Lambda}{p}\right)^{\mathcal{D}/(\mathcal{D}+2)} \qquad (p = \text{const.})$$

These reference values are useful for bosons and fermions.

Universal curves for isochores, isotherms, and isobars:

- p/p_v versus T/T_v at $v = \text{const.}$
- p/p_T versus v/v_T at $T = \text{const.}$
- v/v_p versus T/T_p at $p = \text{const.}$

For fermions we will introduce alternative reference values based on the chemical potential (Fermi energy).

Bose-Einstein condensation [tsl38]

Particles in the gas phase and in the Bose-Einstein condensate (BEC):

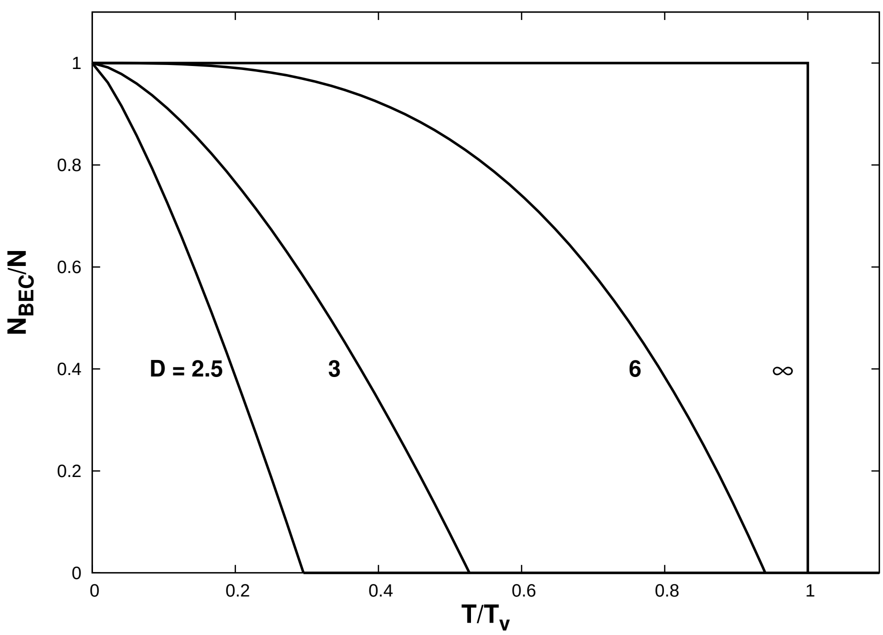
$$\mathcal{N} = \frac{V}{\lambda_T^{\mathcal{D}}} g_{\mathcal{D}/2}(z) + \frac{z}{1-z} = \mathcal{N}_{gas} + \mathcal{N}_{BEC}.$$

Consider process at $v = \text{const}$.
Onset of macroscopic population of the lowest energy level begins when the fugacity locks in to the value $z = 1$:

$$\frac{z}{1-z} = \begin{cases} \mathrm{O}(1), & z < 1, \\ \mathrm{O}(\mathcal{N}), & z = 1. \end{cases}$$

$$T \geq T_c: \quad \frac{\mathcal{N}_{gas}}{\mathcal{N}} = 1, \quad \frac{\mathcal{N}_{BEC}}{\mathcal{N}} = 0.$$

$$T \leq T_c: \quad \begin{cases} \dfrac{\mathcal{N}_{gas}}{\mathcal{N}} = \dfrac{[V/\lambda_T^{\mathcal{D}}]\zeta(\mathcal{D}/2)}{[V/\lambda_{T_c}^{\mathcal{D}}]\zeta(\mathcal{D}/2)} = \left(\dfrac{T}{T_c}\right)^{\mathcal{D}/2}, \\ \dfrac{\mathcal{N}_{BEC}}{\mathcal{N}} = 1 - \dfrac{\mathcal{N}_{gas}}{\mathcal{N}} = 1 - \left(\dfrac{T}{T_c}\right)^{\mathcal{D}/2}. \end{cases}$$



Ideal Bose-Einstein gas: isochores [tsl39]

Isochore at $T \geq T_c$ [tex114]:

$$\frac{p}{p_v} = \frac{g_{\mathcal{D}/2+1}(z)}{\big[g_{\mathcal{D}/2}(z)\big]^{2/\mathcal{D}+1}}, \qquad \frac{T}{T_v} = \big[g_{\mathcal{D}/2}(z)\big]^{-2/\mathcal{D}}.$$

Isochore at $T \leq T_c$ (also valid asymptotically for $T \ll T_v$ in $\mathcal{D} \leq 2$):

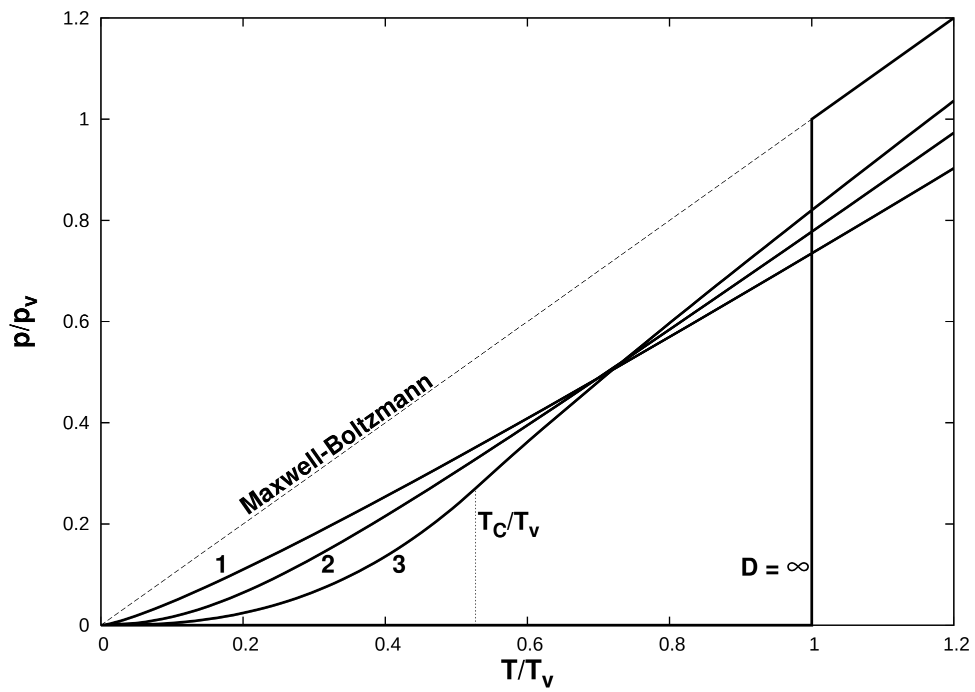
$$\frac{p}{p_v} = \left(\frac{T}{T_v}\right)^{\mathcal{D}/2+1} \zeta(\mathcal{D}/2+1).$$

Critical temperature:

$$\frac{T_c}{T_v} = [\zeta(\mathcal{D}/2)]^{-2/\mathcal{D}} = \begin{cases} 0 & \mathcal{D}=1 \\ 0 & \mathcal{D}=2 \\ 0.527 & \mathcal{D}=3 \\ 1 & \mathcal{D}=\infty \end{cases}$$

High-temperature asymptotic behavior:

$$\frac{p}{p_v} \sim \frac{T}{T_v}\left[1 - \frac{1}{2^{\mathcal{D}/2+1}}\left(\frac{T_v}{T}\right)^{\mathcal{D}/2}\right].$$



[tex114] BE gas in $\mathcal{D}$ dimensions II: isochore

(a) From the fundamental thermodynamic relations for the Bose-Einstein gas in $\mathcal{D}$ dimensions (see [tln67]), derive the following parametric expression for the isochore at $T \geq T_c$:

$$\frac{p}{p_v} = \frac{g_{\mathcal{D}/2+1}(z)}{\big[g_{\mathcal{D}/2}(z)\big]^{2/\mathcal{D}+1}}, \qquad \frac{T}{T_v} = \big[g_{\mathcal{D}/2}(z)\big]^{-2/\mathcal{D}},$$

where $k_B T_v = \Lambda v^{-2/\mathcal{D}}$ and $p_v = \Lambda v^{-2/\mathcal{D}+1}$ with $\Lambda \doteq h^2/2\pi m$ are convenient reference values. (b) Calculate the leading correction to the Maxwell-Boltzmann result at high temperature. (c) Calculate the exact dependence of p/p_v on T/T_v at $T \leq T_c$ in $\mathcal{D} > 2$. Show that this result also holds asymptotically for $T \ll T_v$ in dimensions $\mathcal{D} = 1$ and $\mathcal{D} = 2$.

Solution:

[tex115] BE gas in $\mathcal{D}$ dimensions III: isotherm and isobar

(a) From the fundamental thermodynamic relations for the Bose-Einstein gas in $\mathcal{D} > 2$ dimensions (see [tln67]), derive the following expressions for the isotherm at $v > v_c$ and the isobar at $T \leq T_c$:

$$\frac{p}{p_T} = g_{\mathcal{D}/2+1}(z), \qquad \frac{v}{v_T} = [g_{\mathcal{D}/2}(z)]^{-1};$$

$$\frac{v}{v_p} = \frac{\big[g_{\mathcal{D}/2+1}(z)\big]^{\mathcal{D}/(\mathcal{D}+2)}}{g_{\mathcal{D}/2}(z)}, \qquad \frac{T}{T_p} = \big[g_{\mathcal{D}/2+1}(z)\big]^{-2/(\mathcal{D}+2)}.$$

where $v_T = (\Lambda/k_BT)^{\mathcal{D}/2}$, $p_T = \Lambda(k_BT/\Lambda)^{\mathcal{D}/2+1}$, $k_BT_p = \Lambda(p/\Lambda)^{2/(\mathcal{D}+2)}$, $v_p = (\Lambda/p)^{\mathcal{D}/(\mathcal{D}+2)}$ with $\Lambda \doteq h^2/2\pi m$ are convenient reference values for temperature and pressure and reduced volume. (b) Calculate the leading correction to the Maxwell-Boltzmann result for the isotherm at low density and for the isobar at high temperature.

Solution:

Ideal Bose-Einstein gas: isotherms [tsl40]

For $\mathcal{D} > 2$ we must again distinguish two regimes. At $v > v_c$, all bosons are in the gas phase. At $v < v_c$, a BEC is present. Only the bosons in the gas phase contribute to the pressure.

Isotherm at $v \geq v_c = \lambda_T^{\mathcal{D}}/\zeta(\mathcal{D}/2)$:

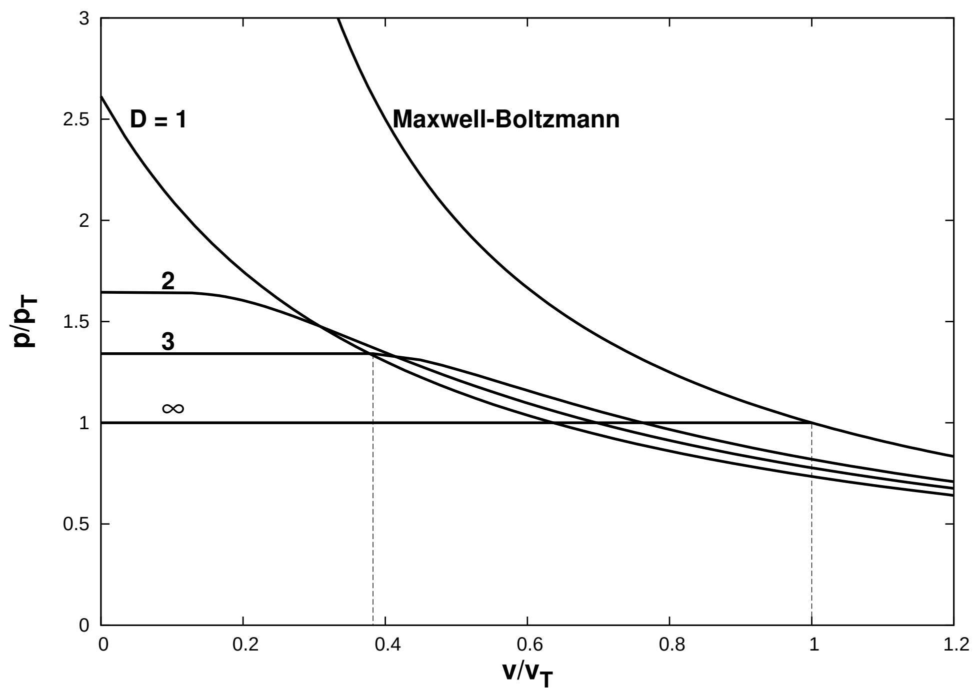
$$\frac{p}{p_T} = g_{\mathcal{D}/2+1}(z), \qquad \frac{v}{v_T} = [g_{\mathcal{D}/2}(z)]^{-1}.$$

Isotherm at $v \leq v_c$:

$$\frac{p}{p_T} = \frac{p_c}{p_T} = \zeta(\mathcal{D}/2+1) = \begin{cases} 2.612 & \mathcal{D} = 1 \\ 1.645 & \mathcal{D} = 2 \\ 1.341 & \mathcal{D} = 3 \\ 1 & \mathcal{D} = \infty \end{cases}$$

Critical (reduced) volume:

$$\frac{v_c}{v_T} = [\zeta(\mathcal{D}/2)]^{-1} = \begin{cases} 0 & \mathcal{D} = 1 \\ 0 & \mathcal{D} = 2 \\ 0.383 & \mathcal{D} = 3 \\ 1 & \mathcal{D} = \infty \end{cases}$$



Ideal Bose-Einstein gas: isobars [tsl48]

A phase transition at $T_c > 0$ takes place in all dimensions $\mathcal{D} \geq 1$. However, the existence of a BEC requires $v_c > 0$, which is realized only for $\mathcal{D} > 2$.

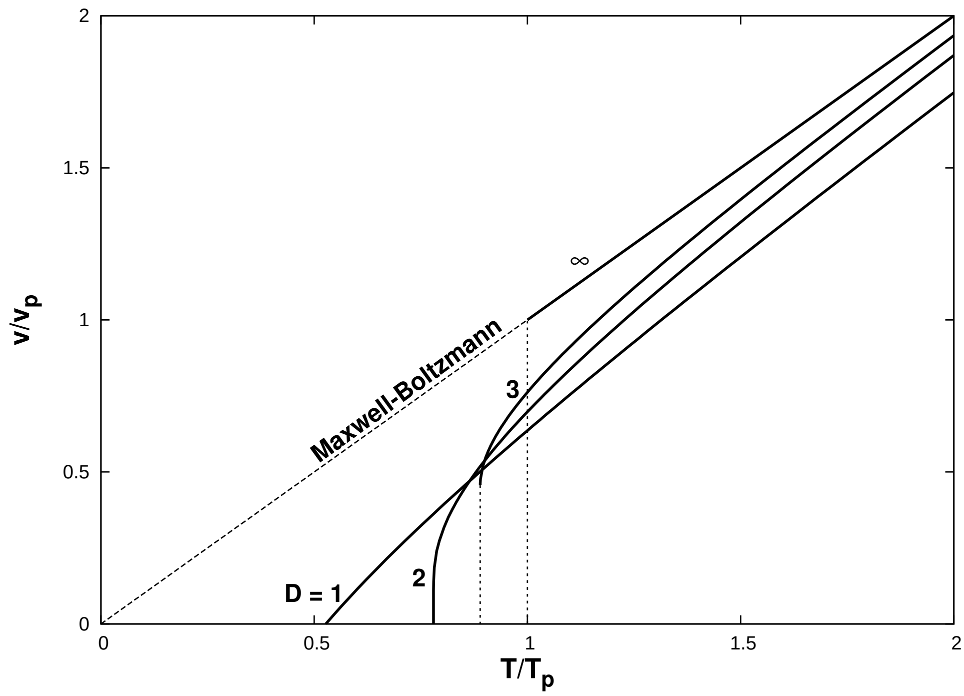
Isobar at $T > T_c$:

$$\frac{v}{v_p} = \frac{\left[g_{\mathcal{D}/2+1}(z)\right]^{\mathcal{D}/(\mathcal{D}+2)}}{g_{\mathcal{D}/2}(z)}, \qquad \frac{T}{T_p} = \left[g_{\mathcal{D}/2+1}(z)\right]^{-2/(\mathcal{D}+2)}.$$

Critical point:

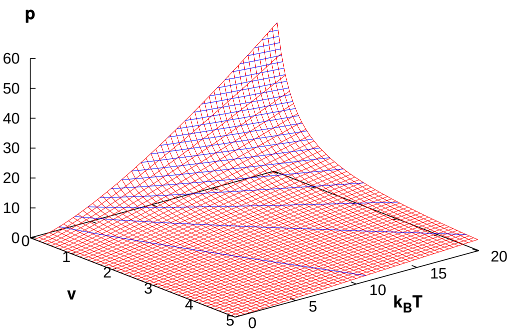
$$\frac{v_c}{v_p} = \frac{[\zeta(\mathcal{D}/2+1)]^{\mathcal{D}/(\mathcal{D}+2)}}{\zeta(\mathcal{D}/2)} = \begin{cases} 0 & \mathcal{D}=1 \\ 0 & \mathcal{D}=2 \\ 0.383 & \mathcal{D}=3 \\ 1 & \mathcal{D}=\infty \end{cases}$$

$$\frac{T_c}{T_p} = [\zeta(\mathcal{D}/2+1)]^{-2/(\mathcal{D}+2)} = \begin{cases} 0.527 & \mathcal{D}=1 \\ 0.779 & \mathcal{D}=2 \\ 0.884 & \mathcal{D}=3 \\ 1 & \mathcal{D}=\infty \end{cases}$$

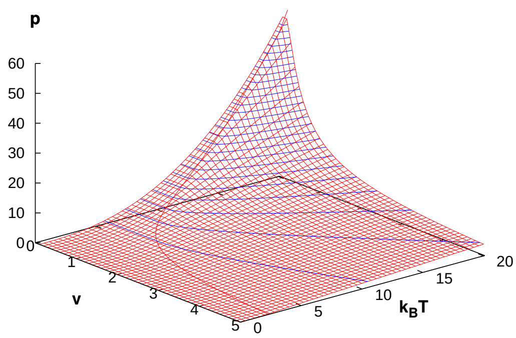


Ideal Bose-Einstein gas: phase diagram [tln72]

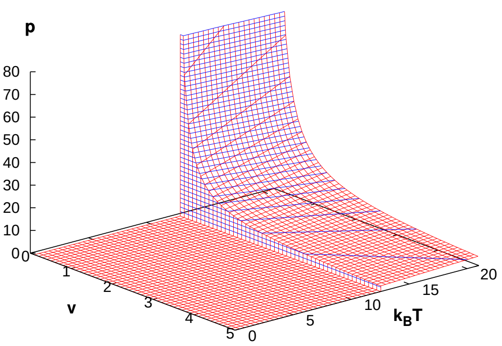
$\mathcal{D} = 1$



$\mathcal{D} = 3$



$\mathcal{D} = \infty$



$$pv = \begin{cases} k_B T, & T > T_c \\ 0, & T < T_c \end{cases}$$

$$k_B T_c = \Lambda \doteq \frac{h^2}{2\pi m}.$$

- $\mathcal{D} = 1$: Transition at $T \geq 0$ and $v = 0$ (transition line = isochore).
- $\mathcal{D} = 3$: Transition at $T > 0$ and $v > 0$.
- $\mathcal{D} = \infty$: Transition at $T > 0$ and $v > 0$ (transition line = isotherm).

Ideal Bose-Einstein gas: heat capacity [tsl41]

Internal energy:

$$\frac{U}{\mathcal{N}k_BT_v} = \begin{cases} \dfrac{\mathcal{D}}{2}\dfrac{g_{\mathcal{D}/2+1}(z)}{g_{\mathcal{D}/2}(z)}\dfrac{T}{T_v}, & T \geq T_c, \\ \dfrac{\mathcal{D}}{2}\zeta(\mathcal{D}/2+1)\left(\dfrac{T}{T_v}\right)^{\mathcal{D}/2+1}, & T \leq T_c. \end{cases}$$

Heat capacity at $T \geq T_c$ [use $zg'_n(z) = g_{n-1}(z)$ for $n \geq 1$]:

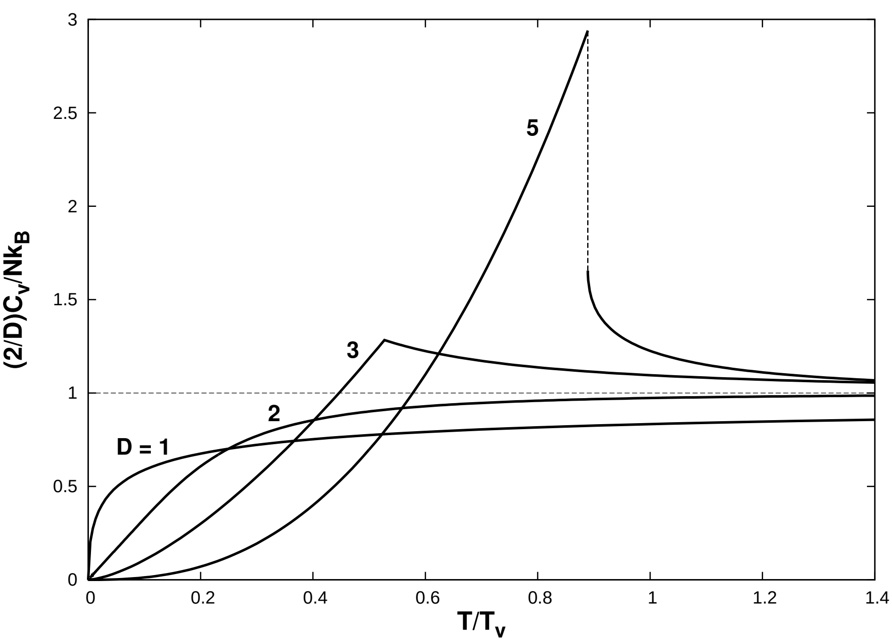
$$\frac{C_V}{\mathcal{N}k_B} = \left(\frac{\mathcal{D}}{2}+\frac{\mathcal{D}^2}{4}\right)\frac{g_{\mathcal{D}/2+1}(z)}{g_{\mathcal{D}/2}(z)} - \frac{\mathcal{D}^2}{4}\frac{g'_{\mathcal{D}/2+1}(z)}{g'_{\mathcal{D}/2}(z)}.$$

Heat capacity at $T \leq T_c$:

$$\frac{C_V}{\mathcal{N}k_B} = \left(\frac{\mathcal{D}}{2}+\frac{\mathcal{D}^2}{4}\right)\zeta\left(\frac{\mathcal{D}}{2}+1\right)\left(\frac{T}{T_v}\right)^{\mathcal{D}/2} = \left(\frac{\mathcal{D}}{2}+\frac{\mathcal{D}^2}{4}\right)\frac{\zeta\left(\frac{\mathcal{D}}{2}+1\right)}{\zeta\left(\frac{\mathcal{D}}{2}\right)}\left(\frac{T}{T_c}\right)^{\mathcal{D}/2}.$$

High-temperature asymptotic behavior:

$$\frac{C_V}{\mathcal{N}k_B} \sim \frac{\mathcal{D}}{2}\left[1+\frac{\mathcal{D}/2-1}{2^{\mathcal{D}/2+1}}\left(\frac{T_v}{T}\right)^{\mathcal{D}/2}\right].$$



[tex97] BE gas in $\mathcal{D}$ dimensions IV: heat capacity at high temperature

The internal energy of the ideal Bose-Einstein gas in $\mathcal{D}$ dimensions and at $T \geq T_c$ is given by the following expression:

$$U = \mathcal{N} k_B T \, \frac{\mathcal{D}}{2} \, \frac{g_{\mathcal{D}/2+1}(z)}{g_{\mathcal{D}/2}(z)}.$$

Use this result to derive the following expression for the heat capacity $C_V = (\partial U/\partial T)_{V\mathcal{N}}$:

$$\frac{C_V}{\mathcal{N} k_B} = \left(\frac{\mathcal{D}}{2} + \frac{\mathcal{D}^2}{4}\right) \frac{g_{\mathcal{D}/2+1}(z)}{g_{\mathcal{D}/2}(z)} - \frac{\mathcal{D}^2}{4} \, \frac{g'_{\mathcal{D}/2+1}(z)}{g'_{\mathcal{D}/2}(z)}.$$

Use the derivative $\partial/\partial T$ of the result $g_{\mathcal{D}/2}(z) = \mathcal{N}\lambda_T^{\mathcal{D}}/V$ with $V = L^{\mathcal{D}}$ to calculate any occurrence of $(\partial z/\partial T)_{V\mathcal{N}}$ in the derivation. Use the recursion relation $z g'_n(z) = g_{n-1}(z)$ for $n \geq 1$ to further simplify the results pertaining to $\mathcal{D} \geq 2$.

Solution:

[tex116] BE gas in $\mathcal{D}$ dimensions V: heat capacity at low temperature

The internal energy of the ideal Bose-Einstein gas in $\mathcal{D} > 2$ dimensions and at $T \leq T_c$ is given by the following expression:

$$\frac{U}{\mathcal{N} k_B T_v} = \frac{\mathcal{D}}{2} \zeta(\mathcal{D}/2 + 1) \left(\frac{T}{T_v}\right)^{\mathcal{D}/2+1}$$

(a) Use this result to derive the following expression for the heat capacity $C_V = (\partial U/\partial T)_{V\mathcal{N}}$:

$$\frac{C_V}{\mathcal{N} k_B} = \left(\frac{\mathcal{D}}{2} + \frac{\mathcal{D}^2}{4}\right) \frac{\zeta\left(\frac{\mathcal{D}}{2}+1\right)}{\zeta\left(\frac{\mathcal{D}}{2}\right)} \left(\frac{T}{T_c}\right)^{\mathcal{D}/2},$$

where $T_c = T_v[\zeta(\mathcal{D}/2)]^{-2/\mathcal{D}}$ is the critical temperature and $k_B T_v = \Lambda/v^{2/\mathcal{D}}$ with $v \doteq V/\mathcal{N}$ and $\Lambda \doteq h^2/2\pi m$ a convenient reference temperature. (b) Show that the heat capacity is continuous at $T = T_c$ if $\mathcal{D} \leq 4$ and discontinuous if $\mathcal{D} > 4$. Find the discontinuity $\Delta C_V/\mathcal{N} k_B$ as a function of $\mathcal{D}$ for $\mathcal{D} > 4$. (c) Infer from the result of [tex97] the leading singularity of $C_V/\mathcal{N} k_B$ at $T/T_v \ll 1$ for $\mathcal{D} = 1$ and $\mathcal{D} = 2$. Then show that these singularitues are consistent with the expression for $C_V/\mathcal{N} k_B$ obtained here in part (a) provided we substitute $(T_v/T_c)^{\mathcal{D}/2} = \zeta(\mathcal{D}/2)$.

Solution:

[tex128] BE gas in $\mathcal{D}$ dimensions VI: isothermal compressibility

(a) Show that the isothermal compressibility, $\kappa_T = -(1/V)(\partial V/\partial p)_{T\mathcal{N}}$, of the ideal BE gas in $\mathcal{D}$ dimensions at $T > T_c$ is

$$p_T\kappa_T = \frac{g'_{\mathcal{D}/2}(z)}{g_{\mathcal{D}/2}(z)g'_{\mathcal{D}/2+1}(z)}, \quad \frac{v}{v_T} = \frac{1}{g_{\mathcal{D}/2}(z)},$$

where $v \doteq V/\mathcal{N}$, $v_T \doteq (\Lambda/k_BT)^{\mathcal{D}/2}$, $p_T \doteq k_BT/v_T$, $\Lambda \doteq h^2/2\pi m$, and $g_n(z)$ are BE functions. Use $zg'_n(z) = g_{n-1}(z)$ for $n \geq 1$ to simplify the results in $\mathcal{D} \geq 2$. (b) Sketch $p_T\kappa_T$ versus v/v_T for $v \geq 0$ in $\mathcal{D} = 1$ and for $v \geq v_c$ in $\mathcal{D} = 3$, where $v_c/v_T = [\zeta(\mathcal{D}/2)]^{-1}$ marks the onset of BEC. (c) Determine the nature of the singularity of κ_T as $v/v_T \to 0$ in $\mathcal{D} = 1, 2$. Determine the critical compressibility $p_T\kappa_T$ at $v = v_c$ in $\mathcal{D} = 3, 5$.

Solution:

[tex129] BE gas in $\mathcal{D}$ dimensions VII: isobaric expansivity

To derive the parametric expression of the isobaric expansivity of the ideal BE gas at $T > T_c$,

$$T_p\alpha_p = \frac{T_p}{T}\left[\left(\frac{\mathcal{D}}{2}+1\right)\frac{g_{\mathcal{D}/2+1}(z)g'_{\mathcal{D}/2}(z)}{g_{\mathcal{D}/2}(z)g'_{\mathcal{D}/2+1}(z)} - \frac{\mathcal{D}}{2}\right], \quad \frac{T_p}{T} = \left[g_{\mathcal{D}/2+1}(z)\right]^{\mathcal{D}/2+1},$$

where $k_BT_p = \Lambda(p/\Lambda)^{2/(\mathcal{D}+2)}$, $\Lambda \doteq h^2/2\pi m$, and $g_n(z)$ are BE functions, establish first the general thermodynamic relation $\alpha_p = \kappa_T(\partial p/\partial T)_v$ with $v \doteq V/\mathcal{N}$, the BE-specific relation $C_V = \mathcal{N}(\mathcal{D}/2)v(\partial p/\partial T)_v$, and the results for C_V and κ_T calculated in [tex97] and [tex128].

Solution:

[tex130] BE gas in $\mathcal{D}$ dimensions VIII: speed of sound

(a) Start from the relation $c = (\rho\kappa_S)^{-1/2}$ for the speed of sound as established in [tex18], where $\rho = m/v$ is the mass density and κ_S the adiabatic compressibility. Use general thermodynamic relations between response functions to derive the following expression for c in terms of dimensionless quantities:

$$\frac{mc^2}{k_BT} = \frac{(v/v_T)}{(p_T\kappa_T)}\left[1 + \frac{(T/T_p)^2(v/v_T)(T_p\alpha_p)^2}{(p_T\kappa_T)(C_V/\mathcal{N}k_B)}\right],$$

where v_T, p_T, T_p are defined in [tln71]. (b) Use the expressions derived in [tex129] for α_p, in [tex128] for κ_T, and in [tex97] for C_V to derive the result

$$\frac{mc^2}{k_BT} = \gamma\frac{g_{\mathcal{D}/2+1}(z)}{g_{\mathcal{D}/2}(z)}, \quad \gamma = 1 + \frac{2}{\mathcal{D}}.$$

(c) Relate the T-dependence of mc^2 to that of the isochore for $v = \text{const}$ and to that of the isobar for $p = \text{const}$.

Solution:

[tex98] Ultrarelativistic Bose−Einstein gas

Consider a Bose-Einstein gas with ultrarelativistic one-particle energy $\epsilon_k = c\hbar k = cp$ in the grandcanonical ensemble at temperature T and chemical potential $\mu = 0$.
(a) Show that the one-particle density of states is $D(\epsilon) = (4\pi V/h^3c^3)\epsilon^2$.
(b) Calculate the pressure $p(T)$, the internal energy $U(T,V)$, and the average number of particles in excited states $\mathcal{N}_\epsilon(T,V)$.
(c) Show that the heat capacity is $C_V/k_B = [16\pi^5/15h^3c^3]V(k_BT)^3$.

Solution:

Blackbody radiation [tln68]

Electromagnetic radiation inside cavity in thermal equilibrium at temperature T. Grandcanonical ensemble of photons ($\epsilon = \hbar\omega = cp$, $\mathbf{p} = \hbar\mathbf{k}$, spin $s = 1$, bosonic, purely transverse).

Density of states: $D(\epsilon) = g\,\dfrac{4\pi V}{h^3c^3}\,\epsilon^2$ with $g = 2$ independent polarizations.

Average occupation number: $\langle n_\epsilon\rangle_{BE} = \dfrac{1}{e^{\beta\epsilon}-1}$.

Number of photons with energies between ϵ and $\epsilon + d\epsilon$:

$$dN(\epsilon) = \langle n_\epsilon\rangle_{BE} D(\epsilon) d\epsilon = \frac{8\pi V\epsilon^2}{h^3c^3}\frac{1}{e^{\beta\epsilon}-1}d\epsilon.$$

Spectral density inside cavity: [use $dN(\epsilon) = V dn(\omega)$ and $\epsilon = \hbar\omega$]:

$$\frac{dn(\omega)}{d\omega} = \frac{\hbar}{V}\frac{dN(\epsilon)}{d\epsilon} = \frac{\omega^2}{\pi^2c^3}\frac{1}{e^{\beta\hbar\omega}-1}.$$

Spectral energy density inside cavity: $du = \hbar\omega dn = \rho(\omega)d\omega$.

$$\rho(\omega)d\omega = \frac{\omega^2}{\pi^2c^3}\frac{\hbar\omega}{e^{\beta\hbar\omega}-1}d\omega = \frac{8\pi\nu^2}{c^3}\frac{h\nu}{e^{\beta h\nu}-1}d\nu.$$

Rate (per unit area) at which particles with (average) speed c escape from cavity through small opening [tex62]: $dN/dt = \frac{1}{4}(N/V)c$.

Spectral density of radiation: $R(\omega) = \dfrac{c}{4}\dfrac{dn(\omega)}{d\omega} = \dfrac{\omega^2}{4\pi^2c^2}\dfrac{1}{e^{\beta\hbar\omega}-1}$.

Spectral energy density of radiation:

$$Q(\omega) = \hbar\omega R(\omega) = \frac{\omega^2}{4\pi^2c^2}\frac{\hbar\omega}{e^{\beta\hbar\omega}-1} \quad \text{(Planck radiation law).}$$

High frequencies: ultrarelativistic MB particles [use $\langle n_\epsilon\rangle_{MB} = e^{-\beta\epsilon}$]:

$$Q(\omega) = \frac{\hbar\omega^3}{4\pi^2c^2}\,e^{-\beta\omega} \quad \text{(Wien radiation law).}$$

Low frequencies: equipartition law applied to electromagnetic modes:

$$Q(\omega) = \frac{k_BT\omega^2}{4\pi^2c^2} \quad \text{(Rayleigh−Jeans radiation law).}$$

[tex105] Statistical mechanics of blackbody radiation

Electromagnetic radiation inside a cavity is in thermal equilibrium with the walls at temperature T. This system can be described by a grandcanonical ensemble (with $\mu = 0$) of photons (massless bosonic particles) with energy $\epsilon = \hbar\omega$ and density of states $\bar{D}(\omega) = (V/\pi^2 c^3)\omega^2$.
(a) Show that the internal energy can be expressed in the form

$$U(T,V) = \sigma V T^4, \quad \sigma = \frac{\pi^2 k_B^4}{15\hbar^3 c^3}$$

as postulated in a previous thermodydnamics problem [tex23].
(b) Show that the equation of state can be expressed in the form $pV = \frac{1}{3}U(T,V)$ as was also postulated in [tex23].

Solution: